

## §8. Riemann-Roch Theorem

Setup:  $C = \text{irr. proj. } \overset{\text{plane}}{\curvearrowright} \text{ curve with nonsingular model}$

$$f: X \rightarrow C. \quad K = k(C) = k(X).$$

$P \in X$ ,  $\text{ord}_P = \text{order function on } K$ .

### §8.1. divisors

(Weier) divisor on  $X$

$$D = \sum_{P \in X} n_P \cdot P$$

$n_P \in \mathbb{Z}$  &  $n_P \geq 0$  for almost all  $P \in X$ .

$$\deg(D) := \sum_{P \in X} n_P \in \mathbb{Z}$$

$$\begin{aligned} F \cdot G &= \sum I(P, F \cdot G) \cdot P \\ \text{if } F &= \text{proj. with} \\ F \cdot G &\text{ divisor on } F \end{aligned}$$

Fact:  $\text{Div}(X)$  the set of all divisors on  $X$  forms an abelian gp.  
free abelian gp on the set  $X$ .

$$\sum n_P \cdot P \geq \sum m_P \cdot P \stackrel{\text{def}}{\iff} n_P \geq m_P \quad \forall P.$$

$$D = \sum n_P \cdot P \stackrel{\text{def}}{=} \text{effective (or, positive)} \iff D \geq 0. \quad (\text{i.e. } n_P \geq 0, \forall P)$$

Example:  $C = \text{plane curve of deg } n.$

$G = \text{plane curve not containing } C \text{ as a component. (of deg } m) \quad mn$

$$\text{div}(G) := \sum_{P \in X} \text{ord}_P(G) \cdot P \in \text{Div}(X) \quad \sum_{P' \in C} I(P', C \cap G) \geq 0$$

Fact:  $\text{div}(G)$  is of deg.  $mn$ . pf:  $\deg(\text{div}(G)) = \sum_{P \in X} \text{ord}_P(G) = \sum_{P' \in C} \left( \sum_{P \in f^{-1}(P')} \text{ord}_P(G) \right)$   
desc desc.

Example: (principal divisors)

$\forall z \in K^*$ , divisor of  $z$  is defined as

$$\text{div}(z) = \sum_{P \in X} \text{ord}_P(z) \cdot P$$

$$\text{ord}_P(z) = 0$$

$$P(X) := \{ \text{div}(z) \mid z \neq 0 \} \subset \text{Div}(X) \text{ subgroup}$$

Basic Facts: 1)  $\text{div}(zz') = \text{div}(z) + \text{div}(z')$

2)  $\text{div}(z^{-1}) = -\text{div}(z)$ ,

3)  $\deg(\text{div}(z)) = 0$   $(z \in K \neq K(C))$

$$z = \frac{F \bmod I(C)}{G \bmod I(C)} \quad \deg F = \deg G.$$

$\text{ord}_P(z) = 0$  for almost  
Simple pt  $P \in C$ .

if not.  
 $\exists$  only pt  $P_1, \dots$   
s.t.  $z(P_i) = 0$

Def: Two divisor  $D, D'$  are called linear equivalent if

$$D' = D + \text{div}(z)$$

for some  $z \in K^*$ . In which case, write  $D' \equiv D$ .

$$\text{div}(z) = \text{div}(F) - \text{div}(G)$$

Weil divisor class group  $CL(X) = \text{Div}(X) / \equiv = \underline{\text{Div}(X)} / \underline{P(X)}$

$$0 \rightarrow K^* \rightarrow K^* \xrightarrow{\text{div}} \text{Div}(X) \rightarrow CL(X) \rightarrow 0$$

$$\text{div}(z) = 0 \Leftrightarrow z \in K^*$$

$$\Leftarrow \vee$$

$$\Rightarrow :$$

$$\text{div}(z) = \text{div}_+(z) - \text{div}_\infty(z)$$

$z$

$$z = \frac{F}{G}$$

$$\text{div}(z) := \sum_{P \in X} \text{ord}_P(z) \cdot P$$

$$\text{div}_+(z) := \sum_{\substack{P \in X \\ \text{ord}_P(z) \geq 0}} \text{ord}_P(z) \cdot P \geq 0$$

$$\text{div}_\infty(z) := \sum_{\substack{P \in X \\ \text{ord}_P(z) < 0}} (-\text{ord}_P(z)) \cdot P \geq 0$$

$$D = \sum_p n_p \cdot p \in \text{Div}(X).$$

$$\begin{aligned} 1^\circ \quad n_p > 0 \quad \text{ord}_p(f) \geq -n_p &\Rightarrow f \\ 2^\circ \quad n_p < 0 \quad \text{ord}_p(f) \geq -n_p &\Rightarrow f \end{aligned}$$

$$L(D) := \{f \in K^* \mid \text{ord}_p(f) \geq -n_p \text{ for all } p \in X\} \cup \{0\}.$$

→ set of rational functions with poles only at the chosen points and with poles no worse than order  $n_p$  at  $p$ .

Fact: 1)  $f \in L(D) \Leftrightarrow \text{div}(f) + D \geq 0$

2)  $L(D)$  forms a v.s. over  $k$ .

$$\ell(D) := \dim_k L(D)$$

$$\begin{cases} f \in L(D) \Rightarrow \gamma f \in L(D) \quad \checkmark \\ f+g \in L(D) \Rightarrow f+g \in L(D) \end{cases}$$

$$\text{ord}_p(f+g) \geq \min(\text{ord}_p(f), \text{ord}_p(g))$$

aim calculate  $\ell(D)$ .

Prop 3. (1).  $D \leq D' \Rightarrow L(D) \subset L(D')$  &  $\dim(L(D')/L(D)) \leq \deg(D'-D)$

(2).  $L(0) = k$ ;  $L(D) = 0$ , if  $\deg(D) < 0$ .

(3).  $\deg(D) \geq 0 \Rightarrow \ell(D) \leq \deg(D) + 1$ .

(4).  $D \equiv D' \Rightarrow \ell(D) = \ell(D')$

pf (1):  $\forall f \in L(D) \Rightarrow \text{div}(f) + D \geq 0 \Rightarrow \text{div}(f) + D' \geq 0 \Rightarrow f \in L(D')$

Assume  $D' = D + p \in D$

$$D' = m p + \sum_{q \neq p} n_q \cdot q$$

$$0 \rightarrow L(D) \rightarrow L(D') \xrightarrow{\varphi} k$$

$$\left(\frac{z}{t}\right) \mapsto \left(\frac{t^m z}{t}\right)_p$$

$$t \mid z t^m$$

$$t \quad \text{ord}_p(t) = i$$

$$z t^m \in \mathcal{O}_p(K)$$

$$\text{ord}_p(z t^m) \geq 0$$

↑

$$\text{div}(z) + D' \geq 0 \Rightarrow \text{ord}_p(z) + m \geq 0 \Rightarrow \text{ord}_p(z) \geq -m$$

$$\begin{aligned} m_p(x) &\rightarrow \mathcal{O}_p(X) \rightarrow k \\ f &\mapsto f(p) \end{aligned}$$

$$\begin{aligned}
 \textcircled{2} \quad \text{div}(f) + D \geq 0 \quad & f \in L(D) \setminus \{0\} \\
 \Rightarrow \text{deg}(\text{div}(f)) + \text{deg}(D) \geq 0 \\
 & \quad \quad \quad \uparrow \\
 \Rightarrow \text{div}(D) \geq 0
 \end{aligned}$$

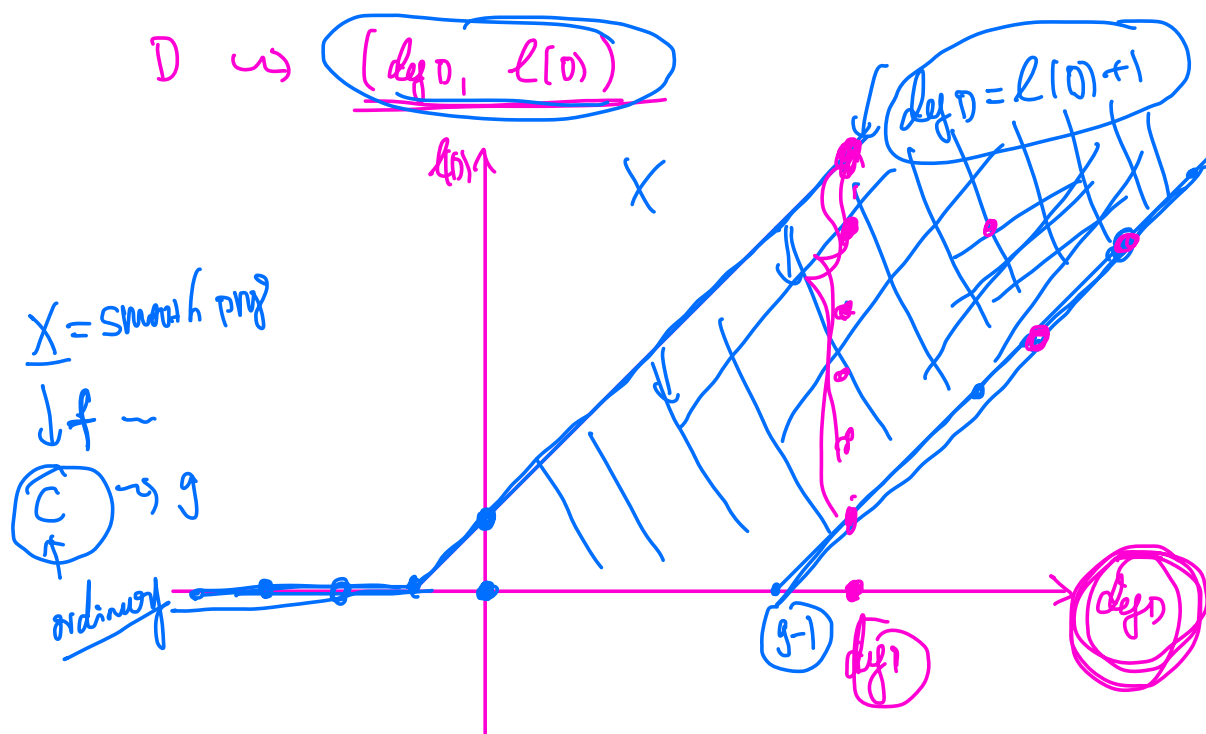
$$D \rightarrow P_1 \rightarrow P_1 + P_2 \rightarrow \dots$$

$$\rightarrow D = P_1 + \dots + P_r$$

$$\begin{array}{ccc}
 L(D) & \rightarrow & L(P_1) \rightarrow \dots \\
 \textcircled{+1} & & +1 \\
 \textcircled{+0} & & +0
 \end{array}$$

$\text{deg } D$

$$\frac{L(D)}{\textcircled{\text{deg } D + 1}} \leq \frac{\text{deg } D + \text{deg}(D)}{\textcircled{\text{deg } D + 1}}$$



Thm (Riemann's thm)  $\exists$   $g$  integer s.t.  $l(D) \geq \deg(D) + 1 - g$ .  
for all  $D$ .

$$g = \max_D \{ \deg D + 1 - l(D) \} \in \{0, 1, 2, \dots\}.$$

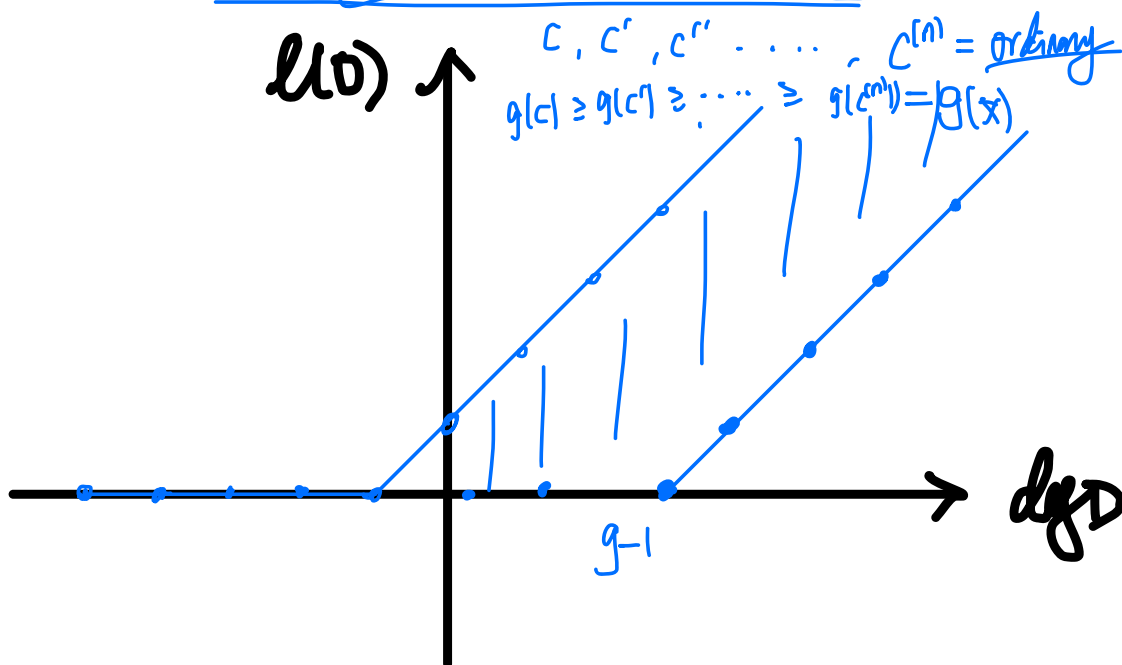
is called the genus of  $X$  (or of  $K$ , or of  $C$ )

Prop  $C =$  plane curve with only ordinary multiple pts.  
 $n = \deg$  of  $C$ ,  $r_P = m_P(C)$ . Then

$$g = \frac{(n-1)(n-2)}{2} - \sum_{P \in C} \frac{r_P(r_P-1)}{2}.$$

Cor 1:  $C =$  plane curve of  $\deg n$ .  $r_P = m_P(C)$ .  $P \in C$ . Then

$$g \leq \frac{(n-1)(n-2)}{2} - \sum \frac{r_P(r_P-1)}{2} \quad g^*(C)$$



## §8.4. Derivation and differentials

# algebraic background to study differentials on a curve

$$df = f_{x_1} dx_1 + f_{x_2} dx_2 + \dots + f_{x_n} dx_n$$

$\mathcal{R}$  = ring containing  $k$ .

$\mathcal{R}$ -mod

$$\sum_{x \in \mathcal{R}} r_x [x] \quad r_x \in \mathcal{R}$$

$$\Omega_k(\mathcal{R}) := F / N \leftarrow \begin{array}{l} \text{submodule of } F \text{ generated} \\ \text{by } \textcircled{1} \textcircled{2} \textcircled{3} \end{array}$$

free  $\mathcal{R}$ -module on set  $\{[x] \mid x \in \mathcal{R}\}$

module of differentials

$$\textcircled{1}: [x+y] - [x] - [y]$$

$$\textcircled{2}: [\lambda x] - \lambda [x]$$

$$\textcircled{3}: [xy] - x[y] - y[x]$$

$$\begin{array}{l} \textcircled{d}: \mathcal{R} \rightarrow \Omega_k(\mathcal{R}) \\ x \mapsto [x] =: dx \end{array}$$

$$[x] \in F \rightarrow F/N$$

$$[x] \mapsto dx$$

$$d(x+y) = [x+y] = [x] + [y] = dx + dy$$

$$d(\lambda x) = \lambda dx$$

$$d(xy) = dx + ydy$$

Fact: 1)  $\mathcal{R} = k[x_1, \dots, x_n] \Rightarrow \Omega_k(\mathcal{R}) = \sum_{i=1}^n \mathcal{R} \cdot dx_i$

2)  $K = k(x_1, \dots, x_n) \Rightarrow \Omega_k(K) = \sum_{i=1}^n K \cdot dx_i$

$$K = k(x, y) \Rightarrow \Omega = k(dx) + k(dy)$$

Prop 1).  $K = \text{alg. function field in one variable over } k$ . Then

$$\Omega_K(K) = \text{1-dim. vect. sp. over } K$$

2). ( $\text{char } k = 0$ ).  $x \in K \setminus k \Rightarrow \Omega_K(K) = K \cdot dx$

$\Rightarrow$  one can define  $\frac{df}{dx}$

$$f(y) = 0$$

$$\frac{F(x,y)=0}{\Downarrow}$$

$$dx \neq 0$$

$$dF(x,y) = 0$$

$$\Rightarrow F_x(x,y)dx + F_y(x,y)dy = 0$$

$$\text{Char } k = p$$

$$(x^p) \in k[k]$$

$$\Rightarrow dx^p = p x^{p-1} = 0$$

$\deg = 0$   
principal divisor

$$\forall z \in K^* \mapsto \text{div}(z)$$

$$w \in \Omega_K(K) \mapsto \text{div}(w)$$

$\deg = 2g - 2$   
canonical divisor

$$\sum (n_p) p$$

## § 8.5 Canonical Divisors.

$$w = f \cdot dt \quad \exists! f \in K$$

$X =$  nonsingular model of a projective curve  $C$ , with function field  $K$

$$\omega \in \Omega = \Omega_K(K). \quad (\omega \in \Omega \text{ is called differential on } X \text{ (or on } C))$$

$$\text{ord}_P(\omega) = \text{ord}_P(f)$$

$$\omega \in \Omega_K(K) = K \cdot dt$$

if  $\omega = f dt$  for some uniformizer  $t$  in  $\mathcal{O}_P(X)$ , = DVR  $(dt \neq 0)$

$$\text{well-defined: } u \sim t \Rightarrow f dt = \omega = g du \Rightarrow f/g = \frac{du}{dt} \in \mathcal{O}^\times \Rightarrow v.$$

$$\text{div}(\omega) := \sum_{P \in X} \text{ord}_P(\omega) \cdot P \in D_{\text{div}}(X) \quad (\omega \neq 0)$$

well-defined (prop 8). a canonical divisor

Fact:  $\omega$  is a canonical divisor. Then

$$\deg(\omega) = 2g - 2$$

Find missing term in Riemann's thm.

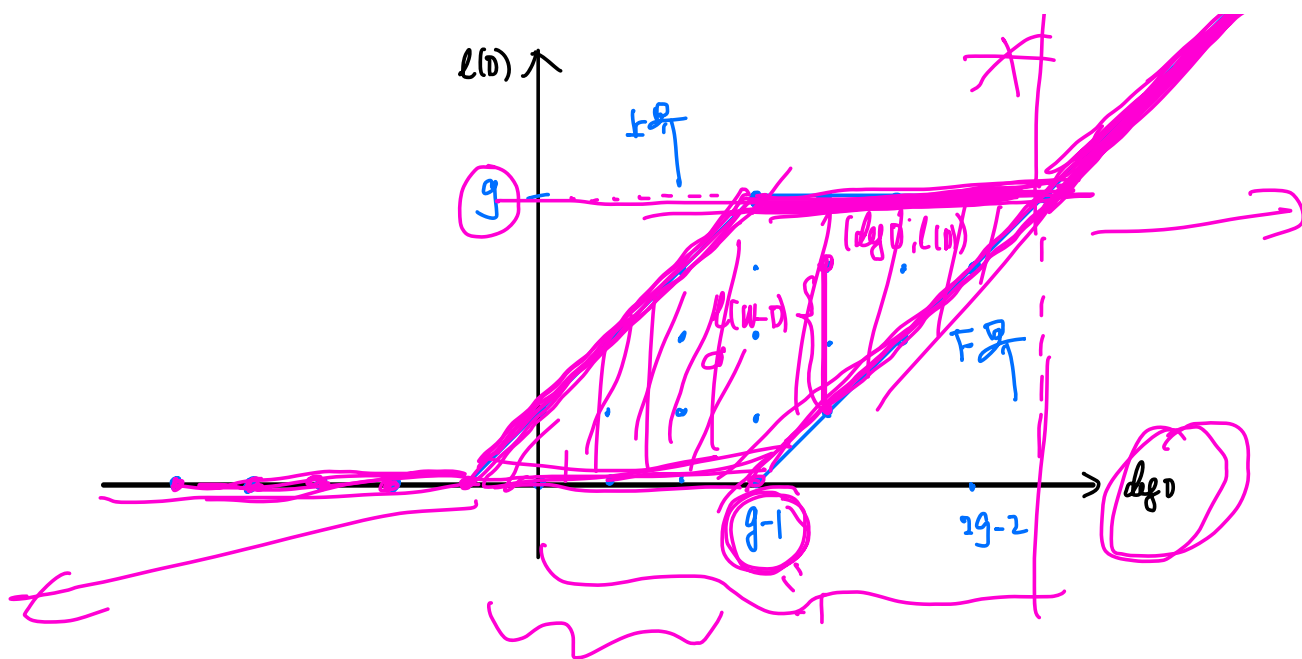
Thm (Riemann-Roch thm)  $\omega$  = canonical divisor on  $X$ . Then

$$l(D) = \deg D + 1 - g + l(\omega - D)$$

Rmk: • holds for  $D \gg 0$  or  $D \ll 0$ .

• compare both sides for  $D$  &  $D+P$ .





1° def  $D \in [0, \dots, g-1]$  ✓✓

2°  $\deg D \in [g, \dots, 2g-2]$  ?

3°  $\deg D \geq 2g-1 \Rightarrow \ell(D) = \deg D + 1 - g$

$$3^o \quad l(D) = \deg D + 1 - g + l(W-D) = 0$$

$$\underline{\deg(W-D)} = \underline{\deg W} - \underline{\deg D}$$

$$= (2g-2) - (2g-1) = -1$$

$$2^0 \quad l(D) = \deg D + 1 - g + l(W-D)$$

$$\deg D \in [g, \dots, 2g-2]$$

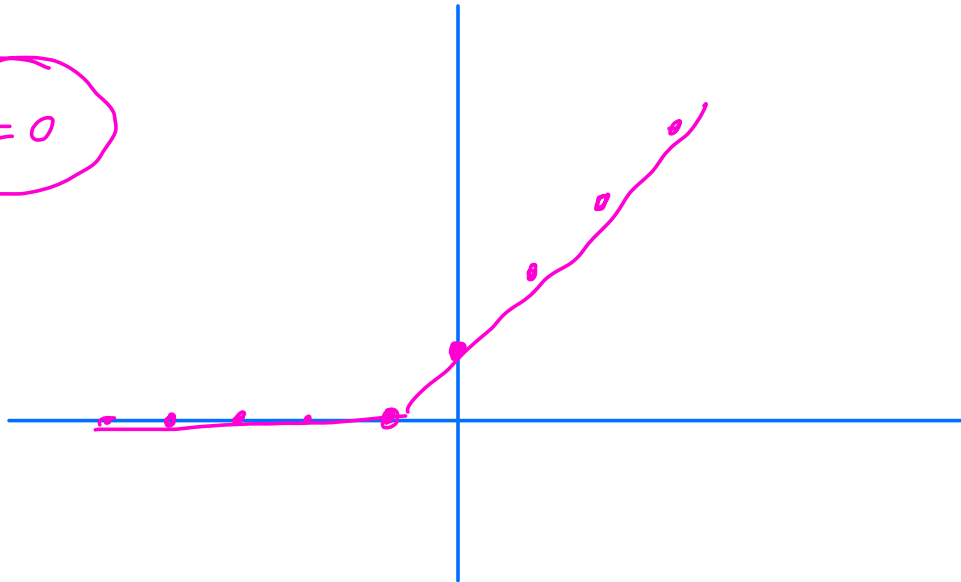
$$\deg(D) + \deg(W-D) = \deg W = 1g-2$$

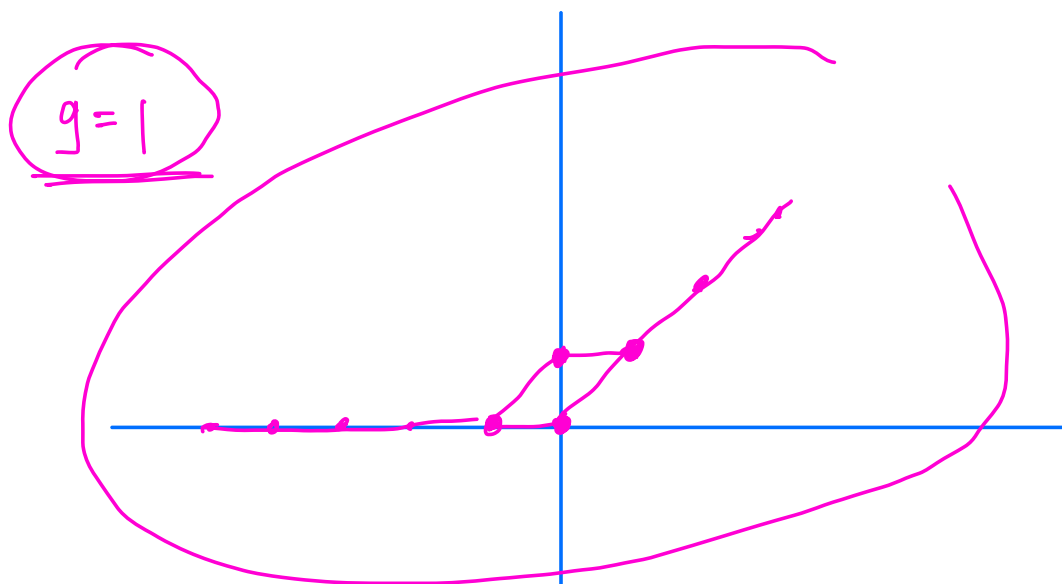
$$\Rightarrow \deg(W-D) = [0, 1, \dots, g-2]$$

$$\Rightarrow l(D) \leq \deg(W-D) + 1$$

$$\begin{aligned} \underline{l(D)} &\leq \underline{\deg D + 1 - g} + \underline{\deg(W-D) + 1} \\ &= 2g-2 + 1 - g = g \end{aligned}$$

$$g=0$$





$$l(0) \rightarrow \textcircled{\text{deg } v}$$

$$\textcircled{\text{deg } v = 0}$$

$$\textcircled{g=1}$$